Int. J. Heat Mass Transfer. Vol. 8, pp. 1–6. Pergamon Press 1965. Printed in Great Britain

AN EXTENSION OF LINEAR MOVING HEAT SOURCE SOLUTIONS TO A TRANSIENT CASE IN A COMPOSITE SYSTEM

GEORGE E. ZINSMEISTER† and JOHN R. DIXON‡
School of Mechanical Engineering, Purdue University, Lafayette, Indiana

(*Received* 22 *April* 1964 *and in revised form* 31 *July* 1964)

Abstract—The transient temperature distribution in a moving bar due to a step input of heat is presented for an insulated and an uninsulated system. It is also shown that this solution is valid for a slender composite system made up of materials of different properties moving independently when the components are of high conductivity and in intimate contact.

NOMENCLATURE

- A, cross-sectional area [ft^2];
- C_p, specific heat [Btu/lb$_m$ degF];
- H, dimensionless parameter defined by equation (10);
- h, convective film coefficient [Btu/h ft degF];
- k, thermal conductivity [Btu/h ft degF];
- Q, total heat input per unit time [Btu/h];
- t, time from start of heating [h];
- U, velocity [ft/h];
- x, distance from heat source in the direction of motion [ft].

Greek symbols

- α, thermal diffusivity, [ft^2/h];
- ρ, mass density [lb$_m$/ft^3];
- θ, temperature [degF].

Superscripts, Subscripts and Indices

- i, refers to an individual component of a composite system;
- n, refers to the total number of components of a composite system;
- $x \geqslant 0$, designates the solutions applicable to the region $x \geqslant 0$;
- $x \leqslant 0$, designates the solutions applicable to the region $x \leqslant 0$;
- $*$, indicates a dimensionless variable;
- $'$, indicates a modified property of a composite system.

INTRODUCTION

SOLUTIONS of the problems of linear heat flow in a bar subjected to a moving heat source have previously been presented for the quasi-stationary state, i.e. a steady state relative to the source (cf. [1], [2]).

These solutions are here extended to the transient case of a step input of heat and to include any number of bars or liquids moving at different velocities in parallel lines through a plane heat source which is considered stationary for the present analysis. See Fig. 1. Any but not all of the velocities may be zero. It is assumed that the temperature at any cross section is uniform across the section and is a function only of the distance from the source at any instant. This requires that the composite system be of relatively small cross section and/or the thermal resistances between the various materials be low. Thus the assumption of uniform temperature at any cross section does not mean neglect of transverse conduction. It results from either the assumption that heat is conducted so rapidly in the transverse direction that no large temperature gradients are able to exist or the assumption that transverse distances are so small compared with those in the direction of motion that large temperature differences can not exist in the

† Graduate Student in Mechanical Engineering.
‡ Associate Professor of Mechanical Engineering.

transverse direction. Radiation effects are neglected in the model under consideration.

This problem arose in the analysis of an instrument for measuring the mass flow of sap in a leaf as originally suggested by Professor A. C. Leopold of the Horticulture Department of Purdue University. As a first approximation, a parallel-veined leaf may be assumed to consist of a series of parallel tubes through which the sap flows and around which are moist non-moving living cells. If, now, one positions a narrow wire heater across the leaf transverse to the direction of flow a temperature field will develop that will change as the mass flow varies. By detecting these temperature changes through properly located thermocouples one can relate them to changes in mass flow via the solution of the problem considered herein.

MATHEMATICAL FORMULATION FOR AN INSULATED SYSTEM

For a differential element of an insulated body of n materials such as that depicted in Fig. 1, there are for each material, heat input terms of the form

$$(-kA)_i \frac{\partial \theta}{\partial x} \Delta t + (\rho C_p A U)_i \theta \, \Delta t,$$

heat output terms of the form

$$(-kA)_i \frac{\partial}{\partial x}\left(\theta + \frac{\partial \theta}{\partial x}\Delta x\right)\Delta t + (\rho C_p A U)_i \left(\theta + \frac{\partial \theta}{\partial x}\Delta x\right)\Delta t,$$

and storage terms of the form

$$(\rho C_p A)_i \, \Delta\theta \, \Delta x$$

where the notation $(\ )_i$ is used to indicate that the product within the brackets is formed using the appropriate quantities of the i^{th} material. For liquids, the bulk velocity is to be used for U. Applying a heat balance and allowing Δx and Δt to approach zero simultaneously, there results,

$$\sum_{i=1}^{n} (kA)_i \frac{\partial^2 \theta}{\partial x^2} - \sum_{i=1}^{n} (\rho C_p A U)_i \frac{\partial \theta}{\partial x} = \sum_{i=1}^{n} (\rho C_p A)_i \frac{\partial \theta}{\partial t}$$

which may be rewritten as

$$\alpha' \frac{\partial^2 \theta}{\partial x^2} - U' \frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial t} \tag{1}$$

where α' is a thermal diffusivity defined by:

$$\alpha' = \frac{\sum_{i=1}^{n} (kA)_i}{\sum_{i=1}^{n} (\rho C_p A)_i} \tag{2}$$

and U' is a velocity defined by:

$$U' = \frac{\sum_{i=1}^{n} (\rho C_p A U)_i}{\sum_{i=1}^{n} (\rho C_p A)_i}. \tag{3}$$

Note that equation (1) is of the same form as that obtained for a single bar moving through a plane heat source. When a composite material

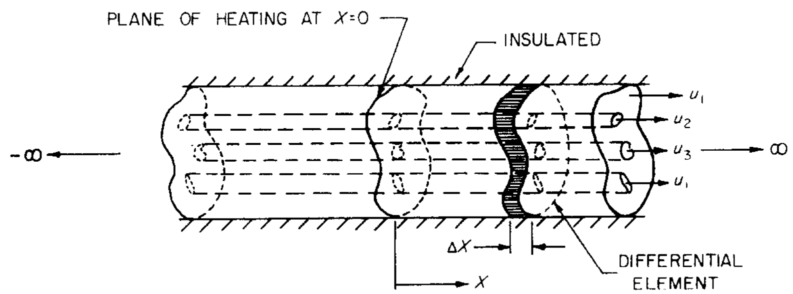

FIG. 1. Composite moving system with plane of heating at $x = 0$.

is considered, the thermal diffusivity and velocity are modified as shown in equations (2) and (3).

We now proceed to extend the solution of (1) to the case of a step input of heat at $t = 0$ when the initial temperature is everywhere zero. The appropriate boundary conditions are

$$\theta = 0 \text{ or } \frac{\partial \theta}{\partial x} = 0 \text{ for } x \to \pm\infty,\ t < \infty \qquad (4)$$

and

$$(kA)'\left(\frac{\partial \theta_{x\leqslant 0}}{\partial x} - \frac{\partial \theta_{x\geqslant 0}}{\partial x}\right) = QH(t) \text{ for } x = 0 \qquad (5)$$

where $(kA)'$ is the quantity $\sum_{i=1}^{n} (kA)_i$ and $H(t)$ is Heaviside's unit step function. $\theta_{x\leqslant 0}$ and $\theta_{x\geqslant 0}$ designate the temperatures for the regions $x \leqslant 0$ and $x \leqslant 0$ respectively. Furthermore, the condition that $\theta_{x\leqslant 0}$ equals $\theta_{x\geqslant 0}$ at $x = 0$ must be satisfied. The boundary condition given by (5) includes one necessary condition in t as well as one in x and is a statement of the fact that the heat introduced at the source is started at t equal to zero and is conducted away in both the positive and negative x directions.

On the introduction of the dimensionless variables

$$t^* = \frac{tu'^2}{\alpha'},\quad x^* = \frac{xu'}{\alpha'},\ \text{and}\ \theta^* = \frac{u'(kA)'\theta}{Q\alpha'},$$

equation (1) and boundary conditions (4) and (5) become

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} - \frac{\partial \theta^*}{\partial x^*} = \frac{\partial \theta^*}{\partial t^*} \qquad (1^*)$$

with

$$\theta^* = 0 \text{ or } \frac{\partial \theta^*}{\partial x^*} = 0 \text{ for } x^* \to \pm\infty,\ t^* < \infty \qquad (4^*)$$

and

$$\frac{\partial \theta^*_{x^*\leqslant 0}}{\partial x^*} - \frac{\partial \theta^*_{x^*\geqslant 0}}{\partial x^*} = H(t^*) \text{ for } x^* = 0 \qquad (5^*)$$

SOLUTION AND RESULTS FOR INSULATED SYSTEM

The solution of the above system of equations was obtained by use of the Laplace transformation (cf. [3], [4]). The mechanics of the solution which were tedious but straight-forward have been placed in the appendix.

For $x^* \leqslant 0$ the result is

$$\theta^*_{x^*\leqslant 0} = \tfrac{1}{2}\left[e^{x^*}\operatorname{erfc}\left(\frac{-x^*}{2\sqrt{t^*}} - \frac{\sqrt{t^*}}{2}\right) - \operatorname{erfc}\left(\frac{-x^*}{2\sqrt{t^*}} + \frac{\sqrt{t^*}}{2}\right)\right] \qquad (6)$$

and for $x^* \geqslant 0$

$$\theta^*_{x^*\geqslant 0} = \tfrac{1}{2}\left[\operatorname{erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \frac{\sqrt{t^*}}{2}\right) - e^{x^*}\operatorname{erfc}\left(\frac{x^*}{2\sqrt{t^*}} + \frac{\sqrt{t^*}}{2}\right)\right]. \qquad (7)$$

If $t^* \to \infty$ in equations (6) and (7), they become the well-known steady state solutions $\theta^*_{x^*\leqslant 0} = e^{x^*}$ and $\theta^*_{x^*\geqslant 0} = 1$. The transient solution is presented graphically in Fig. 2.

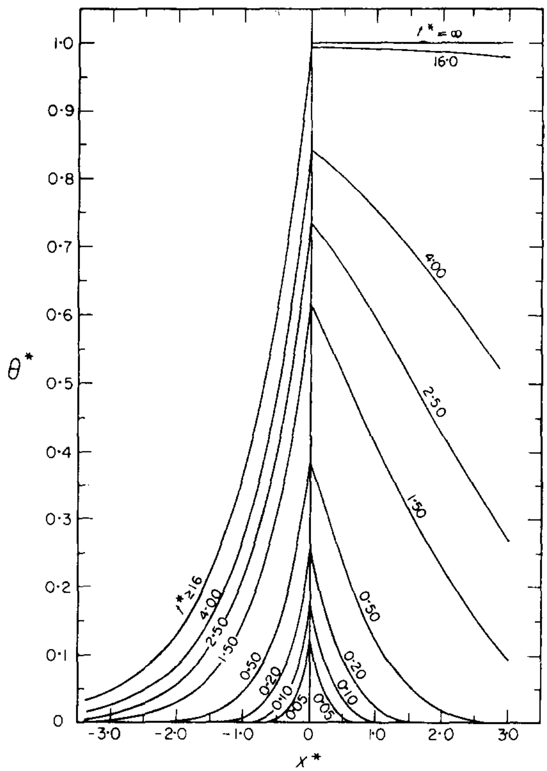

FIG. 2. Temperature distribution about a heat source for a moving insulated system.

MATHEMATICAL FORMULATION FOR A NON-INSULATED SYSTEM

The method of analysis presented above is also applicable to a non-insulated composite system if heat loss is proportional to the difference between the ambient and system temperature at any x location. The constant of proportionality is to be a known convective film coefficient, h.

If it is assumed that the temperature of the environment is zero, then the heat loss to it is $h\theta\Delta x\Delta t$. When this is included in the heat balance the result analogous to equation (1) is

$$\alpha' \frac{\partial^2\theta}{\partial x^2} - U' \frac{\partial\theta}{\partial x} - h'\theta = \frac{\partial\theta}{\partial t} \tag{8}$$

where h' is defined by:

$$h' = \frac{h}{\sum_{i=1}^{n} (\rho c_p A)_i}. \tag{9}$$

If the initial temperature of the composite system is the same as the environment temperature, then the appropriate boundary conditions are again given by equations (4) and (5).

Introducing t^*, x^* and θ^* as previously defined into equation (8) results in

$$\frac{\partial^2\theta^*}{\partial x^{*2}} - \frac{\partial\theta^*}{\partial x^*} - H\theta^* = \frac{\partial\theta^*}{\partial t^*} \tag{8*}$$

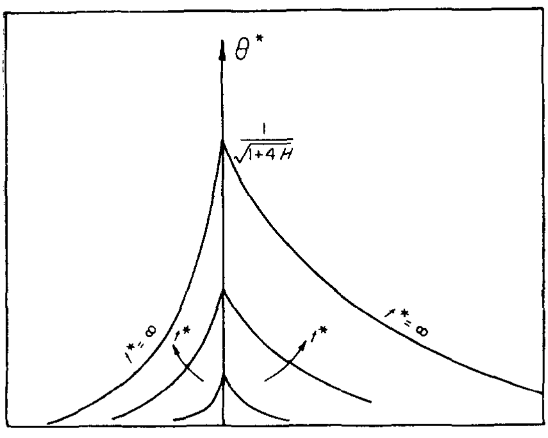

FIG. 3. Temperature distribution about a heat source for a moving non-insulated system for a particular value of H. (Qualitative representation only.)

where H is a dimensionless parameter defined by:

$$H = \frac{h'\alpha'}{U'^2} \tag{10}$$

The dimensionless boundary conditions are again those given by (4*) and (5*).

SOLUTION AND RESULTS FOR A NON-INSULATED SYSTEM

The introduction of the parameter H into the problem complicates the solution only slightly. The Laplace transformation method used is again the same as described in the Appendix. For $x^* \leqslant 0$ the result is

$$\left.\begin{aligned} \theta^*_{x^*\leqslant 0} &= \frac{1}{2\sqrt{(1+4H)}} \Big\{\exp\{\tfrac{1}{2}[1+\sqrt{(1+4H)}]x^*\} \\ &\quad \operatorname{erfc}\left(\frac{-x^*}{2\sqrt{t^*}} - \frac{\sqrt{[(1+4H)t^*]}}{2}\right) \\ &\quad - \exp\{\tfrac{1}{2}[1-\sqrt{(1+4H)}]x^*\} \\ &\quad \operatorname{erfc}\left(\frac{-x^*}{2\sqrt{t^*}} + \frac{\sqrt{[(1+4H)t^*]}}{2}\right)\Big\} \end{aligned}\right\} \tag{11}$$

and for $x^* \geqslant 0$

$$\left.\begin{aligned} \theta^*_{x^*\geqslant 0} &= \frac{1}{2\sqrt{(1+4H)}} \Big\{\exp\{\tfrac{1}{2}[1-\sqrt{(1+4H)}]x^*\} \\ &\quad \operatorname{erfc}\left(\frac{x^*}{2\sqrt{t^*}} - \frac{\sqrt{[(1+4H)t^*]}}{2}\right) \\ &\quad - \exp\{\tfrac{1}{2}[1+\sqrt{(1+4H)}]x^*\} \\ &\quad \operatorname{erfc}\left(\frac{x^*}{2\sqrt{t^*}} + \frac{\sqrt{[(1+4H)t^*]}}{2}\right)\Big\} \end{aligned}\right\} \tag{12}$$

The steady state solutions obtained by allowing $t^* \to \infty$ in (11) and (12) are in agreement with those presented in the literature (cf. [1], [2]). These are

$$\theta^*_{x^*\leqslant 0} = \frac{\exp\{\tfrac{1}{2}[1+\sqrt{(1+4H)}]x^*\}}{\sqrt{(1+4H)}}$$

and

$$\theta^*_{x^*\geqslant 0} = \frac{\exp\{\tfrac{1}{2}[1-\sqrt{(1+4H)}]x^*\}}{\sqrt{(1+4H)}}$$

For a given choice of the parameter H, the graphical representation of equations (11) and (12) will be as shown in Fig. 3. All of the curves in this Figure are only qualitatively correct.

CONCLUSION

The result of this analysis is that the problem of heat transfer in a composite system of materials moving at different velocities through a heat source can, in some instances, be reduced to an analogous situation involving a single bar whose properties are functions of the properties of the composite system. This is true, however, only if it is not inaccurate to assume that the temperature at any cross section is uniform and only a function of the distance from the source at any instant.

The transient solution contained herein should find application in many diversified areas including, for example, the problem of heat distribution in a gun barrel and discontinuous welding and extrusion processes. For starting and stopping in these latter processes, the temperature distribution in the materials can be found from the solutions presented by the method of superposition.

REFERENCES

1. D. Rosenthal, The theory of moving sources of heat and its application to metal treatments, *Trans. Amer. Soc. Mech. Engrs* **68**, 849–866 (1946).
2. Max Jakob, *Heat Transfer*, Vol. 1, pp. 343–348. John Wiley, New York (1959).
3. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd Ed., pp. 297–326, 494–496. Oxford University Press, London (1959).
4. H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, 2nd Ed. Dover, New York (1963).

APPENDIX

The solution of the following system of equations is desired:

$$\frac{\partial^2\theta^*}{\partial x^{*2}} - \frac{\partial\theta^*}{\partial x^*} = \frac{\partial\theta^*}{\partial t^*} \tag{13}$$

with the boundary conditions

$$\theta^* = 0 \text{ or } \frac{\partial\theta^*}{\partial x^*} = 0 \text{ for } x^* \to \pm\infty,\ t^* < \infty \tag{14}$$

$$\frac{\partial\theta^*_{x^*\leqslant 0}}{\partial x^*} - \frac{\partial\theta^*_{x^*\geqslant 0}}{\partial x^*} = H(t) \text{ for } x^* = 0 \tag{15}$$

and

$$\theta^*_{x^*\leqslant 0} = \theta^*_{x^*\geqslant 0} \text{ for } x^* = 0 \tag{16}$$

By employing the Laplace transformation method equation (13) becomes

$$\frac{\partial^2\overline{\theta^*}}{\partial x^{*2}} - \frac{\partial\overline{\theta^*}}{\partial x^*} = s\overline{\theta^*} - \theta^*(x, +0)$$

where $\overline{\theta^*}(x, s)$ is the Laplace transform of $\theta^*(x, t)$. Since $\theta^*(x, t)$ is initially zero this simplifies to

$$\frac{\partial^2\overline{\theta^*}}{\partial x^{*2}} - \frac{\partial\overline{\theta^*}}{\partial x^*} = s\overline{\theta^*} \tag{17}$$

The boundary conditions transform to

$$\overline{\theta^*} = 0 \text{ or } \frac{\partial\overline{\theta^*}}{\partial x^*} = 0 \text{ for } x^* \to \pm\infty \tag{18}$$

$$\frac{\partial\overline{\theta^*}_{x^*\leqslant 0}}{\partial x^*} - \frac{\partial\overline{\theta^*}_{x^*\geqslant 0}}{\partial x^*} = \frac{1}{s} \text{ for } x^* = 0 \tag{19}$$

and

$$\overline{\theta^*}_{x\leqslant 0} = \overline{\theta^*}_{x^*\geqslant 0} \text{ for } x^* = 0 \tag{20}$$

The system of equations (17) through (20) can now be solved for $\overline{\theta^*}_{x^*\leqslant 0}$ and $\overline{\theta^*}_{x^*\geqslant 0}$. The results being

$$\overline{\theta^*}_{x^*\leqslant 0} = \exp\,[x^*/2]\left(\frac{\exp\,[\frac{1}{2}(1+4s)^{\frac{1}{2}}x^*]}{s(1+4s)^{\frac{1}{2}}}\right) \tag{21}$$

and

$$\overline{\theta^*}_{x^*\geqslant 0} = \exp\,[x^*/2]\left(\frac{\exp\,[-\frac{1}{2}(1+4s)^{\frac{1}{2}}x^*]}{s(1+4s)^{\frac{1}{2}}}\right) \tag{22}$$

Note that to return to the time domain one requires the inversion of a form similar to

$$\frac{\exp\,[-a(b+s)^{\frac{1}{2}}]}{s(b+s)^{\frac{1}{2}}}$$

where a and b are independent of s. This inversion does not seem to be presented in any of the available tables. However, it can be derived from an inversion presented in most references ([3] and [4], for example) without recourse to any complex integration necessary if the inversion theorem were used. Note that

$$L^{-1}\left(\frac{\exp\,[-a(b+s)^{\frac{1}{2}}]}{s(b+s)^{\frac{1}{2}}}\right) = \frac{-2}{a}\frac{\partial}{\partial b}\left[L^{-1}\left(\frac{\exp\,[-a(b+s)]^{\frac{1}{2}}}{s}\right)\right]. \tag{23}$$

Here L^{-1} denotes the inversion operation. Now using the property of the Laplace transformation that $L^{-1}[\bar{f}(b+s)] = \exp[-bt]\,L^{-1}[\bar{f}(s)]$ we can write

$$L^{-1}\left(\frac{\exp[-a(b+s)^{\frac{1}{2}}]}{s}\right) = \exp[-bt]\,L^{-1}\left(\frac{\exp[-as^{\frac{1}{2}}]}{s-b}\right). \quad (24)$$

The inversion required in equation (24) is now found from a table to be

$$\left.\begin{aligned} L^{-1}\left(\frac{\exp[-as^{\frac{1}{2}}]}{s-b}\right) &= \tfrac{1}{2}\exp[bt] \\ &\left\{\exp[-a\sqrt{b}]\operatorname{erfc}\left[\frac{a}{2\sqrt{t}} - \sqrt{(bt)}\right]\right. \\ &\left. + \exp[a\sqrt{b}]\operatorname{erfc}\left[\frac{a}{2\sqrt{t}} + \sqrt{(bt)}\right]\right\}. \end{aligned}\right\} \quad (25)$$

Substituting the result of equation (25) into equation (24) and performing the differentiation of (23) results in

$$\left.\begin{aligned} L^{-1}\left(\frac{\exp[-a(b+s)^{\frac{1}{2}}]}{s(b+s)^{\frac{1}{2}}}\right) &= \frac{1}{2\sqrt{b}} \\ &\left\{\exp[-a\sqrt{b}]\operatorname{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{(bt)}\right)\right. \\ &\left. - \exp[a\sqrt{b}]\operatorname{erfc}\left(\frac{a}{2\sqrt{t}} + \sqrt{(bt)}\right)\right\}. \end{aligned}\right\} \quad (26)$$

Application of this inversion to equations (21) and (23) will result in $\theta^*_{x^*\leqslant 0}$ and $\theta^*_{x^*\geqslant 0}$ as given in the text by equations (6) and (7).

The inversion (26) is also the one necessary for the non-insulated system.

Résumé—La distribution de température transitoire dans une barre en mouvement due à un apport brutal de chaleur est présentée pour un système isolé et un système non-isolé.

On montre aussi que, cette solution est valable pour un système élancé composite fait de matériaux de propriétés se déplaçant indépendamment lorsque les composants sont de conductivité élevée et en contact intime.

Zusammenfassung—Die nichtstationäre Temperaturverteilung in einem bewegten Stab, hervorgerufen durch eine schrittweise Wärmezufuhr, wird in dieser Arbeit für ein isoliertes und nichtisoliertes System mitgeteilt. Es wird auch gezeigt, dass diese Lösung gültig ist für ein dünnes zusammengesetztes System, dessen Materialien verschiedene Eigenschaften haben und sich unabhängig voneinander bewegen, sofern die Komponenten eine hohe Wärmeleitfähigkeit besitzen und in guten Kontakt zu einander stehen.